**ABE 307**

**Unsteady State: Laminar Flow in Between Parallel Plates**

Consider a situation where fluid is at rest and the bottom plate starts moving with velocity v0. As t --> ∞, the velocity profile reaches a steady state. This is because the top plate at rest provides a resistance to the flow which balances the force transferred from momentum imparted by the bottom flow over a long time scale. The objective in this problem is to get the velocity profile for the fluid that reflects the transient and steady state profile.

Equation of Motion

∂vx/∂t = γ ∂2vx/∂t2

Boundary Conditions

Initial Condition: vx = 0, t < 0

Boundary Condition 1: vx = v0, y = 0, t > 0

Boundary Condition 2: vx = 0, y = b, t > 0

Dimensionless Variable: Choose dimensionless variables that can change BCs from 0 to 1.

For choosing dimensionless variables you need to choose variables representative of each variable in the original equation such as velocity, distance, time.

Dimensionless variable: Φ = vx/v0

Dimensionless variable: η = y/b

Dimensionless variable: τ = γt/b2

Convert original Equation of Motion in terms of dimensionless variable:

∂vx/∂t = v ∂2vx/∂t2

∂vx/∂t = ∂vx/∂τ \* ∂τ/∂t

∂τ/∂t = γ/b2

∂vx/∂t = γ/b2 \* ∂vx/∂Φ \* ∂Φ/∂τ

∂vx/∂t = v0γ/b2 \* ∂Φ/∂τ

γ ∂2vx/∂t2 = γ ∂/∂y \* ∂vx/∂y

γ ∂2vx/∂t2 = γ ∂/∂y \* [∂vx/∂η \* ∂η/dy]

γ ∂2vx/∂t2 = γ ∂/∂y \* [∂vx/∂η \* 1/b]

γ ∂2vx/∂t2 = γ ∂/∂y \* [∂vx/∂Φ \* ∂Φ/∂η \* 1/b]

γ ∂2vx/∂t2 = γ ∂/∂y \* [∂Φ/∂η \* v0/b]

γ ∂2vx/∂t2 = γv0/b ∂/∂y \* [∂Φ/∂η]

γ ∂2vx/∂t2 = γv0/b ∂/∂η \* ∂η/dy \* [∂Φ/∂η]

γ ∂2vx/∂t2 = γv0/b ∂2Φ/∂η2

v0γ/b2 \* ∂Φ/∂τ = γv0/b ∂2Φ/∂η2

v0γ/b2 \* ∂Φ/∂τ = γv0/b ∂2Φ/∂η2

∂Φ/∂τ = ∂2Φ/∂η2

Change I.C. and Boundary Conditions in form of Dimensionless Variables:

Initial Condition: Φ = 0, τ < 0

Boundary Condition: Φ = 1, η = 0; τ ≥ 0

Boundary Condition: Φ = 0, η = 1; τ > 0

Finding Solutions to the Problem:

We aim to convert the PDE into ODE through use of BCs and form of velocity profile assumed based on expectations about the dependence of velocity profile on certain parameters.

We know that velocity will have two components: steady state and transient state. We can postulate the velocity as

Φ(η,т) = Φ∞(η) - Φt(η,τ)

Let’s first solve for the steady state

Write the Equation of Motion in dimensionless variable form for steady state.

∂Φ/∂τ = ∂2Φ/∂η2

∂Φ∞/∂τ = ∂2Φ∞/∂η2

0 = ∂Φ∞/∂τ

Is this equation a partial differential equation or an ordinary differential equation? Why?

This is a second order ordinary differential equation because there is dependence on only one variable.

Boundary conditions for steady state:

Initial Conditions:

Φ = 1 at η = 0

Φ = 0 at η = 1

Boundary Conditions:

Φ∞ = 1 at η = 0

Φ∞ = 0 at η = 1

Solve the steady state velocity profile equation

0 = ∂2Φ∞/∂η2

Φ∞ = C1η + C2

At η = 0, Φ∞ = 1 --> C2 = 1

At η = 1, Φ∞ = 0 --> C1 = -1

Φ∞ = 1 - η

Finding solution for the transient state velocity profile

Substitute the overall function in the dimensionless form of equation of motion to obtain a differential equation for transient state velocity profile.

∂Φ/∂τ = ∂2Φ/∂η2

-∂Φt/∂τ = -∂2Φt/∂η2

∂Φt/∂τ = ∂2Φt/∂η2

Write the boundary conditions specifically for transient state velocity component

Φt = 0 at η = 0, 1

Φt = Φ∞ at τ = 0

Φ = 1 at η = 0

Φ∞ = 1 at η = 0

Φ = Φ∞ - Φt

This is where the method of separation of variables will be useful. We now postulate the transient state velocity profile to be product of two functions [f(); g()] which are only dependent on one variable independently

Use this new functional form of transient velocity to get differential equation for solving.

Φt = f(η) \* g(τ)

Sub into equation: dg(τ)/dτ = g(τ) \* d2f(η)/dη2

Our goal of converting out partial differential equations to analytically solvable ordinary differential equations has been achieved. From here on, the solution is all about using proper limits and obtaining solution to these ordinary differential equations that satisfy the requirement for velocity profile.

1/g(τ) \* dg(τ)/dτ = 1/f(η) \* d2f(η)/dη2

The only way for them to be equal is if they are equal to a constant.

1/g(τ) \* dg(τ)/dτ = 1/f(η) \* d2f(η)/dη2 = -C2